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INVARIANT SETS FOR A CLASS OF PERTURBED DIFFERENTIAL EQUATIONS OF RETARDED TYPE

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ABSTRACT

Let X be a Banach space and let a, b, q be real numbers such that $a < b$, $q > 0$. Denote by D a locally closed subset of X. A necessary and sufficient condition for the existence of a mild solution $u \in C([a-q, b_1], X)$, $a < b_1 < b$, to the differential equation $du(t)/dt = Au(t) + f(t, u_t)$, such that $u: [a, b_1] \to D$, $u_a = \varphi$ is given. The linear operator A is the generator of a C_0 semigroup $T(t)$, $t \geqq 0$, with $T(t)$ compact for $t > 0$, $f: [a, b) \times C([-q, 0], D_\lambda) \to X$ is continuous and $\varphi \in C([-q, 0], D_\lambda)$ with $\varphi(0) \in D$. D_λ is a neighbourhood of D. Applications to parabolic partial differential equations with retarded argument are given.

1. Introduction

Throughout this paper X is a real or complex Banach space with norm $\| \cdot \|$, $C = C([a, b], X)$ is the Banach space of continuous functions mapping the interval $[a, b]$ into X, with the topology of uniform convergence. The norm of C will be denoted also by $\| \cdot \|$. Given a function $x: [a-q, a+\alpha] \to X$, $\alpha > 0$, define for each $t \in [a, a+\alpha]$ the function $x_t: [-q, 0] \to X$, by $x_t(\theta) = x(t+\theta)$, $\theta \in [-q, 0]$.

Let A be the generator of a strongly continuous semigroup of linear bounded operators $T(t) \in L(X)$, with $\| T(t) \| \leqq 1$, $t \geqq 0$.

Consider the initial value problem

$$\frac{du(t)}{dt} = Au(t) + f(t, u_t), \qquad a \leqq t < b \tag{1.1}$$

$$u_a = \varphi, \qquad \varphi \in C([-q, 0], X), \tag{1.2}$$

where $u_a(\theta) = u(a+\theta)$, $-q \leqq \theta \leqq 0$.

Following Browder [2], we call a mild solution of (1.1) + (1.2), a solution of the integral equation

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$$u(t) = T(t-a)\varphi(0) + \int_a^t T(t-s)f(s, u_s)\,ds, \qquad a \leqq t < b \tag{1.3}$$

$$u_a = \varphi. \tag{1.4}$$

Let $D \subset X$ be a locally closed subset (i.e. for each $x \in D$ there is $r > 0$ such that $D \cap \{y \in X, \|y - x\| \leqq r\}$ is closed in X). Denote by $D_\lambda = \{y \in X, d(y; D) \leqq \lambda\}$, where λ is an arbitrary positive number and $d(y; D)$ means the distance from y to D.

We say that D is invariant (or forward invariant) set for (1.3) if for each $\varphi \in C([-q, 0], D_\lambda)$, with $\varphi(0) \in D$, there is $c = c(\varphi) > a$ and a solution $u \in C([a-q, c], X)$ to (1.3) such that $u_a = \varphi$ and $u(t) \in D$ for all $t \in [a, c]$, $c < b$.

The main result of this paper is the following local existence theorem:

THEOREM 1.1. *Let $D \subset X$ be a locally closed subset and let $f\colon [a, b) \times C([-q, 0], D_\lambda) \to X$ be a continuous function, $a < b \leqq +\infty$. Let $T(t)$, $t \geqq 0$, be a C_0 semigroup on X. Assume the following condition holds:*

$$\lim_{h \downarrow 0} \frac{1}{h} d(T(h)v(0) + hf(t, v); D) = 0 \tag{1.5}$$

for all $t \in [a, b)$ and $v \in C([-q, 0], D_\lambda)$, with $v(0) \in D$.

If $T(t)$ is compact for $t > 0$, then (1.5) is a necessary and sufficient condition for the existence of a solution $u \in C([a-q, c], X)$ to (1.3) such that $u_a = \varphi$ and $u(t) \in D$ for $t \in [a, c]$, where $c = c(\varphi) \in (a, b)$, $\varphi \in C([-q, 0], D_\lambda)$.

Inasmuch as $|d(x; D) - d(y; D)| \leqq \|x - y\|$, $\forall x, y \in X$, it follows that (1.5) is equivalent to

$$\lim_{h \downarrow 0} \frac{1}{h} d\left(T(h)v(0) + \int_t^{t+h} T(t+h-s)f(t, v)\,ds;\, D\right) = 0 \tag{1.6}$$

for all $t \in [a, b)$ and $v \in C([-q, 0]; D_\lambda)$, with $v(0) \in D$.

The idea of assuming the compactness of $T(t)$ for $t > 0$ in the theory of abstract differential equations, which are characterized by the fact that the associated homogeneous linear problem generates $T(t)$, is due to Pazy [11].

This paper is a generalization of the paper [14] to the case of linear perturbation of differential equations of retarded type.

For proving Theorem 1.1, we construct first of all a sequence of approximate solutions using the techniques of Martin [7], [8], [9] and of Webb [16], under the form developed in [12] and [14]. In the proof of the convergence of these approximate solutions we use the technique of Pazy [11] (as in [14]).

If $q = 0$, we may consider in (1.5) the set D instead of D_λ and thus we obtain the main result of [14], which extends those of Crandall [4] and Pazy [11].

In the case X-finite dimensional we can take $A = 0$ in (1.1). In this case, for $D = X$, (1.5) is automatically satisfied and therefore Theorem 1.1 becomes a well known result of Hale [5].

In [15] no compactness of $T(t)$ is assumed, but restrictions on f are imposed (namely $f(t, v)$ is Lipschitz continuous with respect to v).

In the last part of the paper the continuation of solutions is studied and an application to parabolic partial differential equations with retarded argument is given.

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2. The main results

The main result of this paper is that given by Theorem 1.1. Another result (on the continuation of the solutions of (1.3)) will be given at the end of this section.

PROOF OF THEOREM 1.1

Necessity. Let t be arbitrary in $[a, b)$ and let $v \in C([-q, 0], D_\lambda)$ with $v(0) \in D$. Assume that there is a number $\alpha = \alpha(v) > 0$ and a continuous function u on $[t-q, t+\alpha]$, $t + \alpha < b$, such that

$$(2.1)\quad u(t') = T(t'-t)v(0) + \int_t^{t'} T(t'-s)f(s, u_s)\,ds, \qquad t \leqq t' \leqq t+\alpha$$

$$(2.2)\quad u_t = v \quad \text{on} \quad [-q, 0], \quad u: [t, t+\alpha] \to D,$$

i.e. $u(t+h) \in D$ for all $h \in [0, \alpha]$.

Set $t' - t = h$. Then (2.1) and (2.2) yield

$$\frac{1}{h} d(T(h)v(0) + hf(t, v); D) \leqq \frac{1}{h} \| T(h)v(0) + hf(t, v) - u(t+h) \|$$

$$= \left\| f(t, v) - \frac{1}{h} \int_t^{t+h} T(t+h-s)f(s, u_s)\,ds \right\| \to 0 \quad \text{as} \quad h \to 0.$$

which proves the necessity of (1.5).

Sufficiency. Let φ be an arbitrary element of $C([-q, 0]; D_\lambda)$ with $\varphi(0) \in D$. We will prove that there is $\alpha_1 = \alpha_1(\varphi) > 0$, $\alpha_1 < b - a$, and a continuous function u satisfying (1.3) on $[a, a+\alpha_1]$, $u(t) \in D$ for $t \in [a, a+\alpha_1]$ and $u_a = \varphi$ on $[-q, 0]$. Denote by $S(\varphi(0), r) \subset X$ (respectively $\bar{S}(\varphi, r) \subset C([-q, 0], X)$) the ball of radius r about $\varphi(0) \in D$ (respectively $\varphi \in C([-q,0],X)$). Let $\alpha > 0$ and $r \in (0,\lambda]$ be small enough such that

(2.3) $\|f(t,v)\| \leqq \bar{M}$, for all $t \in [a, a+\alpha]$, $a+\alpha < b$ and $v \in \bar{S}(\varphi, r)$

$$D \cap S(\varphi(0), r) \quad \text{is closed.} \tag{2.4}$$

Such $\alpha > 0$, $r > 0$, and $\bar{M} > 0$ satisfying (2.3) and (2.4) exist since f is continuous (therefore locally bounded) and D is locally closed. Let $r_1 \in (0,r)$. Choose $\alpha_1 \in (0,\alpha]$ such that

(2.5) $\|\varphi(\theta_1) - \varphi(\theta_2)\| < r - r_1$ for $|\theta_1 - \theta_2| \leqq \alpha_1$, $\theta_1, \theta_2 \in [-q, 0]$

$$\max_{0 \leqq t \leqq \alpha_1} \|T(t)\varphi(0) - \varphi(0)\| + \alpha_1 M \leqq r_1, \quad M = 1 + \bar{M}. \tag{2.6}$$

Let n be an arbitrary natural number.

Define a function u^n: $[a-q, a]$, by $u^n(a+\theta) = \varphi(\theta)$, for $\theta \in [-q, 0]$.

According to the definition of x_t given in the Introduction, this means $u^n_a = \varphi$. Define also $t^n_0 = a$, $u^n_0 = \varphi(0) \in D$. Therefore

$$u^n_0 = u^n(t^n_0) = u^n(a) = \varphi(0) \in D \quad \text{and} \quad u^n_{t_0} = \varphi.$$

Assume that we have constructed u^n on $[a, t^n_i]$ with $u^n(t^n_i) = u^n_i \in D \cap S(\varphi(0), r)$, $u^n_{t^n_i} \in C([-q,0], D_\lambda) \cap \bar{S}(\varphi, r)$.

If $t^n_i < a + \alpha$ choose the largest number $d^n_i \in (0, 1/n]$ with the properties

$$t^n_{i+1} = t^n_i + d^n_i \leqq a + \alpha_1 \tag{2.7}$$

(2.8) $\|f(t,v) - f(t, u^n_{t^n_i})\| \leqq 1/n$ for all $t \in [t^n_i, t^n_i + d^n_i]$ and $v \in C([q,0], D_\lambda)$

such that

$$\|v - u^n_{t^n_i}\| \leqq d^n_i M + \max_{0 \leqq t \leqq d^n_i} \|T(t)u^n_i - u^n_i\|,$$

$$\frac{1}{d_i} d\Big(T(d_i)u_i + \int_{t_i}^{t_i+d_i} T(t_i + d_i - s) f(t_i, u_{t_i})\, ds; D\Big) \leqq 1/2n \tag{2.9}$$

where $t_i = t^n_i$, $d_i = d^n_i$ and $u_i = u^n_i$ (i.e., when there is no danger of confusion, we drop n). Also $u_{t^n_i} = u_{t_i} = u^n_{t^n_i}$. By (2.9) we see that there is an element $u_{i+1} \in D$ such that

$$\frac{1}{d_i} \Big\| T(d_i)u_i + \int_{t_i}^{t_i+d_i} T(t_i + d_i - s) f(t_i, u_{t_i})\, ds - u_{i+1} \Big\| \leqq 1/n.$$

Denoting

$$p_i = \frac{1}{d_i}\left(u_{i+1} - T(d_i)u_i - \int_{t_i}^{t_{i+1}} T(t_{i+1}-s)f(t_i, u_{t_i})\,ds\right)$$

we have

$$u_{i+1} = T(t_{i+1}-t_i)u_i + \int_{t_i}^{t_{i+1}} T(t_{i+1}-s)f(t_i, u_{t_i})\,ds + (t_{i+1}-t_i)p_i \tag{2.10}$$

with $\|p_i\| \leqq 1/n$.

Define u^n on $[t_i, t_{i+1}]$ by

$$u^n(t) = T(t-t_i)u_i + \int_{t_i}^{t} T(t-s)f(t_i, u_{t_i})\,ds + (t-t_i)p_i, \quad t_i \leqq t \leqq t_{i+1}. \tag{2.11}$$

Clearly

$$u^n(t_i) = u_i \in D, \qquad u^n(t_{i+1}) = u_{i+1} \in D.$$

On the other hand u^n can be written in the form

$$\begin{aligned} u^n(t) = {} & T(t-a)\varphi(0) + \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} T(t-s)f(t_j, u_{t_j})\,ds + \int_{t_i}^{t} T(t-s)f(t_i, u_{t_i})\,ds \\ & + \sum_{j=0}^{i-1} (t_{j+1}-t_j)T(t-t_{j+1})p_j + (t-t_i)p_i, \qquad t_i \leqq t \leqq t_{i+1} \end{aligned} \tag{2.12}$$

$$u^n(a+\theta) = \varphi(\theta), \qquad -q \leqq \theta \leqq 0, \qquad \text{i.e.} \quad u^n_a = \varphi. \tag{2.13}$$

The proof of this simple fact is left to the reader.

If $t \in [t_i, t_{i+1}]$, (2.12) yields

$$\|u^n(t) - \varphi(0)\| \leqq \|T(t-a)\varphi(0) - \varphi(0)\| + (t-a)M, \qquad t - a < \alpha_1,$$

which implies (in view of (2.6)) $u^n(t) \in S(\varphi(0), r)$, so $u_{i+1} = u^n(t_{i+1}) \in D \cap S(\varphi(0), r)$.

We can prove that $u_{t_{i+1}} \in \bar{S}(\varphi, r)$. For this fact we have to estimate $\|u_{t_{i+1}}(\theta) - \varphi(\theta)\|$ for each $\theta \in [-q, 0]$.

If $-q \leqq \theta \leqq a - t_{i+1}$, then

$$\begin{aligned} \|u_{t_{i+1}}(\theta) - \varphi(\theta)\| &= \|u^n(a + \theta + t_{i+1} - a) - \varphi(\theta)\| \\ &= \|\varphi(t_{i+1} + \theta - a) - \varphi(\theta)\| < r - r_1 < r \end{aligned}$$

since $t_{i+1} - a < \alpha_1$ (so we apply (2.5)).

If $a - t_{i+1} < \theta \leqq 0$, then $t_{i+1} + \theta > a$, $t_{i+1} + \theta - a < t_{i+1} - a < \alpha_1$, $|\theta| < t_{i+1} - a < \alpha_1$. In this case (2.12), (2.5) and (2.6) yield:

$$\|u^n_{t_{i+1}}(\theta)-\varphi(\theta)\| = \|u^n(t_{i+1}+\theta)-\varphi(\theta)\| \leqq \|T(t_{i+1}+\theta-a)\varphi(0)-\varphi(0)\|$$
$$+\alpha_1 M + \|\varphi(\theta)-\varphi(0)\| r_1 + r - r_1 = r.$$

Therefore, for each $\theta \in [-q,0]$ we have $\|u_{t_{i+1}}(\theta)-\varphi(\theta)\| \leqq r$ so $u_{t_{i+1}} \in \bar{S}(\varphi, r)$.

Furthermore $u_{t_{i+1}} \in C([-q,0], D_\lambda)$. Indeed, if $-q \leqq \theta \leqq a - t_{i+1}$, we have already seen that $u_{t_{i+1}}(\theta) = \varphi(t_{i+1}+\theta-a) \in D_\lambda$ (by hypothesis). If $a - t_{i+1} \leqq \theta \leqq 0$, then $u_{t_{i+1}}(\theta) = u^n(t_{i+1}+\theta) \in S(\varphi(0), r) \subset D_\lambda$, and therefore $u_{t_{i+1}}(\theta) \in D_\lambda$ for all $\theta \in [-q, 0]$.

We will show now that $\lim_{i\to\infty} t_i = a + \alpha_1$ and $\lim_{i\to\infty} u_i = u^*$ exists, too.

First of all, since $t_i \leqq a + \alpha_1$, $i = 1, 2, \cdots$, $\lim_{i\to\infty} t_i = l$ exists.

Let $j > i$. Taking into account (2.12) and $u^n(t_i) = u_i$, for $t_i \geqq a$, we have

$$\|u_i - u_j\| \leqq \|T(t_j - t_i)\varphi(0) - \varphi(0)\| + \sum_{m=0}^{i-1} \|(T(t_j - t_i) - I) f(t_m, u_{t_m})\|$$

(2.14)

$$(t_{m+1} - t_m) + \sum_{m=0}^{i-1} \|T(t_j - t_i)p_m - p_m\| (t_{m+1} - t_m) + M(t_j - t_i).$$

Choose k_ε large enough such that:

$$t_j - t_i < \varepsilon'/6M, \quad \|T(t_j - t_i)\varphi(0) - \varphi(0)\| \leqq \varepsilon'/6,$$

(2.15)

$$\forall j \geqq i \geqq k_\varepsilon, \quad \varepsilon = 2\varepsilon' > 0.$$

Choose $\bar{k}_\varepsilon \geqq k_\varepsilon$ large enough such that

(2.16) $\displaystyle \|T(t_j - t_i)p_m - p_m\| \leqq \frac{\varepsilon'}{3(a+\alpha_1)}, \quad m = 0, 1, \cdots, k_\varepsilon - 1, \quad j \geqq i \geqq \bar{k}_\varepsilon,$

(2.17) $\displaystyle \|(T(t_j - t_i) - I) f(t_m, u_{t_m})\| \leqq \frac{\varepsilon'}{3(a+\alpha_1)}, \quad m = 0, 1, \cdots, k_\varepsilon - 1, \quad j \geqq i \geqq \bar{k}_\varepsilon.$

Then we have

$$\sum_{m=0}^{i-1} \|T(t_j - t_i)p_m - p_m\| (t_{m+1} - t_m) \leqq \frac{t_{\bar{k}_\varepsilon}\varepsilon'}{3(a+\alpha_1)} + (2/n)(t_i - t_{\bar{k}_\varepsilon})$$

(2.18)

$$\leqq \frac{\varepsilon'}{3} + \frac{\varepsilon'}{3nM} \leqq \frac{\varepsilon'}{2}$$

since $t_{\bar{k}_\varepsilon} < a + \alpha_1$, $t_i - t_{\bar{k}_\varepsilon} < \varepsilon'/6M$,

$$\|T(t_j - t_i)p_m - p_m\| \leqq 2\|p_m\| \leqq \frac{2}{n}, \quad \frac{1}{3nM} \leqq \frac{1}{6} \quad \text{for} \quad n \geqq 2, \quad M > 1$$

$$\sum_{m=0}^{i-1} \|(T(t_j - t_i) - I) f(t_m, u_{t_m})\| (t_{m+1} - t_m)$$

(2.19)

$$\leqq \frac{\varepsilon' t_{k_\varepsilon}}{3(a+\alpha_1)} + 2(M-1)(t_i - t_{\bar{k}_\varepsilon}) \leqq \frac{2\varepsilon'}{3}$$

since $a < t_{\bar{k}_\varepsilon} \leqq a + \alpha_1$.

Combining (2.15), (2.18), (2.19) and (2.14) we get

$$\|u_i - u_j\| \leqq \frac{\varepsilon'}{6} + \frac{\varepsilon'}{6} + \frac{\varepsilon'}{2} + \frac{2\varepsilon'}{3} = \frac{3\varepsilon'}{2} < 2\varepsilon' = \varepsilon, \quad \forall j \geqq i \geqq \bar{k}_\varepsilon, \tag{2.20}$$

which shows that u_i is a Cauchy sequence, therefore $\lim_{i\to\infty} u_i = u^*$ exists and $u^* \in D \cap S(\varphi(0), r)$.

Define $u^n(l) = u^*$, so u^n is defined on $[a-q, l]$ with values in D_λ. It follows that $\lim_{i\to\infty} u^n_{t_i} = u^n_l$ in $C([-q,0], D_\lambda)$. Indeed $u^n_{t_i}(\theta) = u^n(t_i + \theta) \in D_\lambda$ and $u^n(t_i + \theta) \to u^n(l+\theta)$ as $i \to \infty$, uniformly with respect to $\theta \in [-q, 0]$ (since $u^n : [a-q, l] \to D_\lambda$ is continuous).

Assume by contradiction that $l < a + \alpha_1$. The continuity of f implies the existence of a positive number $\bar{c}$ such that

$$\|f(t,v) - f(l, u^n_l)\| \leqq \frac{1}{3n}, \qquad \text{for all} \quad |t - l| \leqq 2\bar{c}, \quad \|v - u^n_l\| \leqq \bar{c}(M+2). \tag{2.21}$$

Choose s with the properties

$$0 < s < \min(1/n, \bar{c}, a + \alpha_1 - l), \tag{2.22}$$

$$d\left(T(s)u^* + \int_l^{l+s} T(l+s-\tau) f(l, u^n_l)\, d\tau; D\right) \leqq \frac{s}{3n} \tag{2.23}$$

(which is possible, since $u^n_l(0) = u^n(l) = u^*$ and so we may use (1.6) for $t = l$, $v = u^n_l$).

$$\max_{0 \leqq t \leqq s} \|T(t)z - z\| \leqq \bar{c}, \qquad \forall z \in \{u_0, u_1, \cdots, u_i, \cdots\}. \tag{2.24}$$

(The inequality (2.24) is possible since the set $\{u_i\}_{i=0}^\infty$ is relatively compact in X).

Let m be large enough such that $l - t_i < s$, $\|u^n_{t_i} - u^n_l\| \leqq \bar{c}$ (which implies $\|u_i - u^*\| \leqq \bar{c}$, since $u_{t_i}(0) = u^n(t_i) = u_i$, $u^n_l(0) = u^n(l) = u^*$) for all $i \geqq m$. If $(t, v) \in [a, a+l] \times C([-q, 0], D_\lambda)$ is such that $|t - t_i| \leqq s$, $\|v - u_{t_i}\| \leqq sM + \max_{0\leqq t \leqq s} \|T(t)u_i - u_i\|$, then $|t - l| \leqq |t - t_i| + |t_i - l| \leqq 2\bar{c}$, $\|u^n_l - v\| \leqq \|u^n_l - u^n_{t_i}\| + \|u^n_{t_i} - v\| \leqq 2\bar{c} + \bar{c}M$. Therefore, by (2.21) we have

$$\begin{aligned} \|f(t,v) - f(t_i, u_{t_i})\| &\leqq \|f(t,v) - f(l, u_l)\| + \|f(l, u_l) - f(t_i, u_{t_i})\| \\ &\leqq \frac{1}{3n} + \frac{1}{3n} < \frac{1}{n}, \qquad \forall i \geqq m. \end{aligned} \tag{2.25}$$

On the other hand $d_i = t_{i+1} - t_i < l - t_i < s$. Taking into account that d_i is the maximal number in $(0, 1/n]$ satisfying (2.7), (2.8) and (2.9) it follows that

$$(2.26)\quad d\left(T(s)u_i + \int_{t_i}^{t_i+s} T(t_i + s - \tau)f(t_i, u_{t_i})\,d\tau;\, D\right) > \frac{s}{2n}, \quad \text{for all} \quad i \geqq m.$$

Letting $i \to \infty$ in (2.26), in view of (2.23) we get a contradiction, therefore $\lim_{i\to\infty} t_i = l = a + \alpha_1$.

Let us introduce the following two functions:

$$(2.27)\qquad a_n(s) = t_i, \qquad s \in [t_i, t_{i+1}), \qquad a_n(a + \alpha_1) = a + \alpha_1$$

$$(2.28)\qquad g_n(t) = \sum_{j=0}^{i-1} (t_{j+1} - t_j)T(t - t_{j+1})p_j + (t - t_i)p_i, \quad t_i \leqq t \leqq t_{i+1}$$

$g_n(a + \alpha_1) = \lim_{i\to\infty} g_n(t_i)$ (which exists in view of (2.14) and (2.18)).

Then $\|g_n(t)\| \leqq t/n$, for all $a \leqq t \leqq a + \alpha_1$.

It follows that u^n given by (2.12) can be written in the following form:

$$u^n(t) = T(t-a)\varphi(0) + \int_a^t T(t-s)f(a_n(s), u_{a_n(s)})as + g_n(t) \quad a \leqq t \leqq a + \alpha_1$$

$$(2.29)\qquad u^n(a + \theta) = \varphi(\theta), \qquad -q \leqq \theta \leqq 0.$$

Denote

$$(2.30)\qquad y^n(t) = \int_a^t T(t-s)f(a_n(s), u^n_{a_n(s)})\,ds, \quad a \leqq t \leqq a + \alpha_1.$$

Clearly y^n satisfies

$$(2.31)\qquad \|y^n(t)\| \leqq (M-1)\alpha_1,$$

$$(2.32)\quad \|y^n(t) - y^n(\tau)\| \leqq (t-\tau)(M-1) + (M-1)\int_a^\tau \|T(t-s) - T(\tau - s)\|\,ds,$$

for all $t, \tau \in [a, a + \alpha_1]$, $t \geqq \tau$.

Let $t > a$, and $\delta \in (0, t-a)$. Define

$$(2.33)\qquad \begin{aligned} y^n_\delta(t) &= \int_a^{t-\delta} T(t-s)f(a_n(s), u_{a_n(s)})\,ds \\ &= T(\delta)\int_a^{t-\delta} T(t-s-\delta)f(a_n(s), u_{a_n(s)})\,ds. \end{aligned}$$

Since $T(\delta)$ is compact and $\int_a^{t-\delta} T(t-s-\delta)f(a_n(s), u_{a_n(s)})\,ds$ is bounded in n, it follows that $\{y^n_\delta(t)\}_{n=1}^\infty$ is precompact in X. Inasmuch as $\|y^n(t) - y^n_\delta(t)\| \leqq \delta(M-1)$, $\forall \delta \in (0, t-a)$, it follows that $\{y^n(t)\}_{n=1}^\infty$ is also precompact in X. By the Arzela–Ascoli theorem we may assume that $\lim_{n\to\infty} y^n(t) = y(t)$ exists (uniformly on $[a, a + \alpha_1]$). By (2.29) it follows that $\mathrm{Lim}_{n\to\infty} u^n(t) =$

$T(t-a)\varphi(0)+y(t)=u(t)$ exists, $a \leqq t \leqq a+\alpha_1$. Since $a_n(s) \to s$ as $n \to \infty$, we can easily see that $u^n_{a_n(s)} \to u_s$ as $n \to \infty$. Passing to the limit as $n \to \infty$ in (2.29) it follows that $u(t)=T(t-a)\varphi(0)+y(t)$ satisfies (1.3) and (1.4). The fact that $u(t) \in D$ for each $t \in [a, a+\alpha_1]$ follows from the fact that $u^n(a_n(s))=u^n(t_i)=u_i \in D \cap S(\varphi(0), r)$ for $t_i \leqq s \leqq t_{i+1}$. $a_n(s) \to s$ as $n \to \infty$ yields

$$\lim_{n \to \infty} u^n(a_n(s)) = u(s) \in D \cap S(\varphi(0), r).$$

With $c = a + \alpha_1$, the proof of Theorem 1.1 is complete.

3. Continuation of solutions. Examples

Theorem 1.1 is a local existence result of a mild solution to (1.3). In connection with the continuation of solutions on $[a, b)$ the following result holds.

THEOREM 3.1. *Let $f: [a,\infty) \times C([-q,0], D_\lambda) \to X$ be a continuous function which maps bounded sets in $[a,\infty) \times C([-q,0], D_\lambda)$ into bounded sets of X. Let $T(t)$, $t \geqq 0$ be a C_0 semigroup with $T(t)$ compact for $t>0$, such that* (1.5) *is satisfied for all $t \geqq a$ and $v \in C([-q,0], D_\lambda)$ with $v(0) \in D$. Then for each $\varphi \in C([-q,0], X)$,* (1.3) *has a solution on a maximal interval of existence $u: [a, t_{\max}) \to D$, $u_\alpha = \varphi$, where either $t_{\max} = +\infty$, or if $t_{\max} < \infty$, then $\overline{\lim}_{t \to t_{\max}} \|u(t)\| = +\infty$.*

REMARK 3.1. The proof of this theorem is similar to the proof of Theorem 3.1 of Pazy [11], so we omit it. The only new fact the reader has to observe is that the boundedness of u on $[a, t_{\max})$ implies the boundedness of $\{u_t, t \in [a, t_{\max})\}$ in $C([-q,0], X)$.

Let us discuss now the particular case of differential equations with retarded argument of the form

$$\frac{du(t)}{dt} = Au(t) + f(t, u(t-q)), \quad a \leqq t < b, \quad q > 0 \tag{3.1}$$

$$u(a+\theta) = \varphi(\theta), \quad -q \leqq \theta \leqq 0, \quad \varphi \in C([-q,0], X). \tag{3.2}$$

A mild solution to (3.1) is a continuous function $u: [a-q, b) \to X$ satisfying the integral equation

$$u(t) = T(t-a)\varphi(0) + \int_a^t T(t-s) f(s, u(s-q))\, ds, \quad a \leqq t < b. \tag{3.3}$$

Using the notations and the proof of Theorem 1.1 (with minor modifications) we can prove the following result:

THEOREM 3.2. a) *Assume that D and $T(t)$ are as in Theorem* 1.1 *and* $f: [a,b)\times D \to X$ *is continuous. A necessary and sufficient condition for the existence of a mild solution* $u: [a-q, a+\alpha] \to X$ *to* (3.1), *such that* $u_a = \varphi$, $u(t) \in D$, $a \leqq t \leqq a+\alpha$ *is the following*:

$$\lim_{h \downarrow 0} \frac{1}{h} d(T(h)v(0) + hf(t, v(-q)); D) = 0 \tag{3.4}$$

for all $t \in [a,b)$, $v \in C([-q,0], D_\lambda)$ *with* $v(0) \in D$.

b) *If in addition f maps bounded subset of* $[a,b) \times D_\lambda$ *into bounded subsets of X then for* (3.3) *the assertion of Theorem* 3.1 *also holds.*

If D is open then (3.4) is automatically satisfied.

The above results can be applied to partial differential equations with retarded argument. We will give an example in this direction (similar to those given in [14]).

Let Ω be a bounded domain in R^n, $n \geqq 1$, with smooth boundary. Let $X = L^2(\Omega)$ and let $f: R_+ \times R \to R$ be a continuous function such that the Nemytskii operator $F: R_+ \times L^2(\Omega) \to L^2(\Omega)$ induced by f (i.e. $(F(t,u))(x) = f(t, u(x))$, $t \in R_+$, $u \in L^2(\Omega)$) is continuous). It is known that the operator $A = \Delta$, with $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$, generates a C_0 semigroup $T(t)$ with $T(t)$ compact for $t > 0$ (see [14]).

Applying Theorem 3.2 it follows that, for each $C([-q,0]; L^2(\Omega))$, the parabolic partial differential equation with retarded argument

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + f(t, u(t-q,x)), \quad t \geqq 0, \quad x \in \Omega \tag{3.5}$$

$$u(\theta, x) = \varphi(\theta, x), \quad -q \leqq \theta \leqq 0, \quad x \in \Omega \tag{3.6}$$

has a local mild solution.

Indeed, using the above notations and $(u(t))(x) = u(t,x)$ for $u(t) \in L^2(\Omega)$, (3.5) and (3.6) can be written under the form

$$u'(t) = \Delta u(t) + F(t, u(t-q)), \quad t \geqq 0 \tag{3.7}$$

$$u(\theta) = \varphi(\theta), \quad -q \leqq \theta \leqq 0. \tag{3.8}$$

By Theorem 3.2 (with D open), (3.7) + (3.8) has a local mild solution $u: [-q, c(\varphi)] \to L^2(\Omega)$, $c(\varphi) > 0$, with $u(\theta) = \varphi(\theta)$ on $[-q, 0]$.

By definition of F we have $(F(t, u(t-q)))(x) = f(t, u(t-q,x))$, $x \in \Omega$, so $u(t,x)$ is a mild solution of (3.5) + (3.6).

REMARK 3.2. If f doesn't assure the continuity condition of the Nemytskii operator F, then the problem of the existence of a mild solution to (3.5) + (3.6) remains open, because in this case we cannot apply Theorem 3.2 to (3.7) + (3.8). It seems that (3.5) + (3.6) must be treated directly in this case, using a version of a technique of Hess [6].

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